Measure Spaces:

Let X be some set. X is a measure space if $\exists$ a σ-ring $\mathcal{M}$ of subsets of X (called measurable sets) & a non-negative countably additive set function μ (called a measure) defined on $\mathcal{M}$.

If $X \in \mathcal{M}$, then X is called a measurable space.

Example: $X = \mathbb{R}^p$, $\mathcal{M}$ = collection of all Lebesgue measurable sets in $\mathbb{R}^p$, μ = Lebesgue measure.

Example: $X = \{1, 2, 3, \ldots\}$, $\mathcal{M} = P(X)$

$$\mu : E \in \mathcal{M} \longrightarrow |E|$$

Often we refer to $(X, \mathcal{M}, \mu)$

Measurable Functions

We say a function f, defined on a measurable space $\mathcal{M}$ is measurable if

$\{x \mid f(x) > a\}$ is measurable $\forall a$.

i.e. belongs to the σ-ring $\mathcal{M}$.

Example: $X = \mathbb{R}^p$, $\mathcal{M} = \mathcal{M}(\mu)$ = set of μ-measurable sets.

Then every continuous function is measurable because $\{x \mid f(x) > a\}$ is open $\forall a \in \mathbb{R}$.

Theorem: The following are equivalent.

$\{x \mid f(x) > a\}$ is measurable $\forall$ real a
$\{x \mid f(x) \geq a\}$ is measurable $\forall$ real a
$\{x \mid f(x) < a\}$ is measurable $\forall$ real a
$\{x \mid f(x) \leq a\}$ is measurable $\forall$ real a

Proof:

Suppose that $\{x \mid f(x) > a\}$ is measurable $\forall$ real a

But $\{x \mid f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > a - \frac{1}{n}\}$

$\Rightarrow \{x \mid f(x) \geq a\} \in \mathcal{M}$ (countable union)

$\Rightarrow \{x \mid f(x) < a\} = X \setminus \{x \mid f(x) \geq a\} \in \mathcal{M}$

(Set diff, $X \in \mathcal{M}$ by defn of measurable space)

$\Rightarrow \{x \mid f(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) < a + \frac{1}{n}\}$

$\Rightarrow \{x \mid f(x) > a\} = X \setminus \{x \mid f(x) \leq a\} \in \mathcal{M}$. ∎

Goal: Show that ordinary operations like taking limits, absolute values, adding or multiplying functions, applied to measurable f$^{+}_{-}$s, give measurable f$^{+}_{-}$s

Theorem: If f is measurable then $|f|$ is measurable too.

Proof: f measurable $\Rightarrow \{x \mid f(x) > a\} \in \mathcal{M}$

& $\{x \mid f(x) < -a\} \in \mathcal{M}$ $\forall a \in \mathbb{R}$

$\Rightarrow \{x \mid |f(x)| > a\} \in \mathcal{M} \Rightarrow |f|$ is measurable

Theorem: Functions If $\{f_n\}$ is a sequence of measurable functions

and $g(x) := \inf_n f_n(x) \quad \forall x \in X$

and $h(x) := \lim_{n\to\infty} \sup_{n} f_n(x)$ (sup then inf)

Then g & h are measurable

Proof: Need to show that

$$\{x \mid g(x) > a\} \in \mathcal{M}$$

but $\{x \mid g(x) > a\} = \{x \mid \sup f_n(x) > a\}$

$$= \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\}$$

$\in \mathcal{M} \Rightarrow g$ is measurable (so is the inf)

Now, Let $g_m(x) = \sup_{n \geq m} f_n(x)$. This is measurable

then $h(x) = \inf_m g_m(x)$ so this is measurable

∎

Corollaries:

f & g are measurable $\Rightarrow$

(i) $\max(f,g)$ & $\min(f,g)$ are measurable

(ii) $f^+ = \max(f,0)$ & $f^- = -\min(f,0)$ are measurable.

(iii) The limit of a converging sequence of measurable functions is measurable.

Theorem: Let f & g be measurable real valued functions on X.

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be continuous

and set $h(x) = F(f(x), g(x)) \quad \forall x \in X$

Then h is measurable

(example: fg, $f+g$)

Proof: need $\{x \mid F(f(x), g(x)) > a\} \in \mathcal{M}$

Let $G_a = \{(u,v) \mid F(u,v) > a\}$

and note that G_a is open (why?)

$\Rightarrow G_a = \bigcup_{n=1}^{\infty} I_n \longleftarrow$ sequence of open intervals in $\mathbb{R}^2$

$= \bigcup_{n=1}^{\infty} (a_n, b_n) \times (c_n, d_n)$

But $\{x \mid a_n < f(x) < b_n\} \in \mathcal{M}$ (why?)

$\{x \mid c_n < g(x) < d_n\} \in \mathcal{M}$

$\Rightarrow \underbrace{\{x \mid a_n < f(x) < b_n, c_n < g(x) < d_n\}}_{I_n} \in \mathcal{M}$

$\Rightarrow G_a \in \mathcal{M} \quad \forall a \in \mathbb{R}.$

Remark: Defining a measure was not necessary for discussing measurable functions. (we only need measurable sets, i.e., sets in the σ-ring $\mathcal{M}$)

Example: Borel-measurable functions on $\mathbb{R}^p$ are functions for which $\{x \mid F(x) > a\}$ is a Borel set.

Simple Functions

Def'n (Characteristic function)

Let $E \subset X$, then $K_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$

is the characteristic function of E.

Def'n: A simple function is a real valued function on X with finite range.

meaning $|\text{range}| < \infty$

- Every simple function is a finite linear combination of characteristic function

~~Proof~~: Let $c_1 \dots c_n$ be the range of s

and ~~Let~~ let $E_i = \{x : s(x) = c_i\}$

then $s(x) = \sum_{i=1}^{n} c_i K_{E_i}(x)$

As a consequence: s is measurable function iff the sets E_i are all measurable

Theorem (Every function can be approximated by simple functions)

Let $f: X \to \mathbb{R}$, $\exists$ a sequence of simple functions s_n, such that

$$s_n(x) \to f(x) \text{ as } n \to \infty \quad \forall x \in X.$$

If f is measurable, each s_n can be chosen to be measurable. If $f \geq 0$, $\{s_n\}$ may be chosen monotonically increasing

Proof: Let's start with non-negative functions

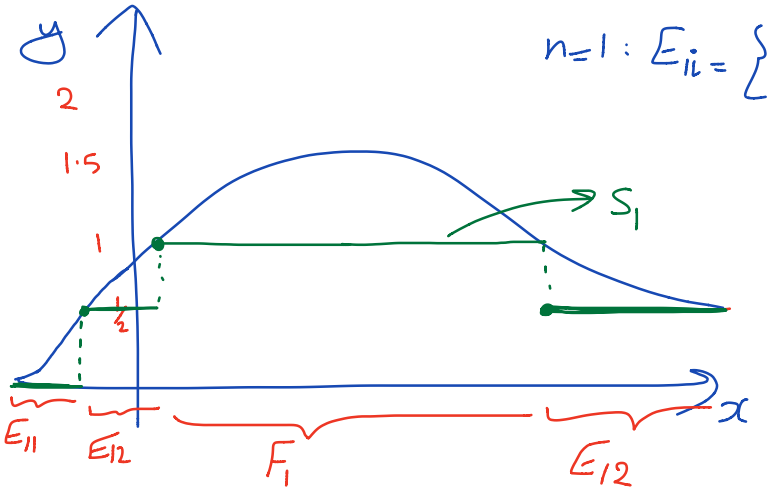
For $f \geq 0$ define

$$E_{ni} = \left\{ x \;\middle|\; \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\}$$

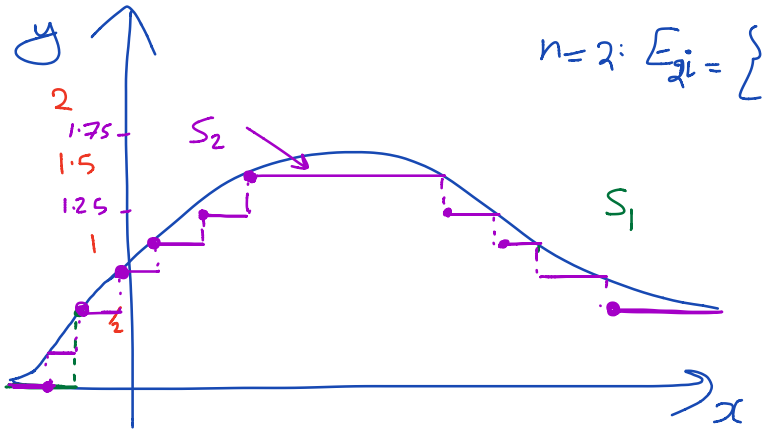
& $F_n = \{ x \mid f(x) \geq n \}$

For $n = 1, 2, 3, \ldots$ & $i = 1, 2, 3, \ldots, n2^n$

Let $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} K_{E_{ni}} + nK_{F_n}$



$n=1: E_{1i} = \left\{ x \mid \frac{i-1}{2} \le f(x) < i/2 \right\}$



$n=2: E_{2i} = \left\{ x \mid \frac{i-1}{2^2} \le f(x) < i/4 \right\}$

When f is not ≥ 0

write $f = f^+ - f^-$ and do the same const. as above for each.

$\{s_n\} \longrightarrow f$

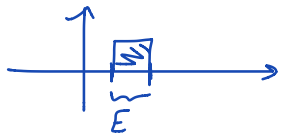
Integration:

We will work with measurables spaces

$(X, \mathcal{M}, \mu)$

- **Characteristic functions.** If $E \in \mathcal{M}$

$$\int_E K_E \, d\mu = \mu(E)$$



(matches our intuition, the integral is $\mu(E) \cdot 1$ = base · height)

for area in the case $X = \mathbb{R}$, $\mu = m$

- **Simple functions**

Suppose $S(x) = \sum_{i=1}^{n} c_i K_{E_i}(x)$ with $c_i \geq 0$

is measurable (so $E_i \in \mathcal{M}$)

and suppose $E \in \mathcal{M}$

$$\int_E S \, d\mu = \sum_{i=1}^{n} c_i \, \mu(E \cap E_i)$$

- Non-negative functions

Suppose f is measurable

$$\int_E f\,d\mu = \sup_{s \text{ simple}, s \le f} \sum_{i=1}^{n} c_i \mu(E \cap E_i)$$

↖ Lebesgue integral of f wrt the measure μ over the set E.
Note that $\int_E f\,d\mu$ may be $+\infty$

- Measurable functions

Let f be measurable and suppose

$$\int_E f^+ d\mu < \infty \quad \text{or} \quad \int_E f^- d\mu < \infty$$

we define $$\int_E f\,d\mu = \int_E f^+ d\mu + \int_E f^- d\mu$$

If $\int_E f^+ d\mu$ & $\int_E f^- d\mu$ are finite, then $\int_E f\,d\mu$ is finite and we say f is Lebesgue integrable on E.

Notation: $f \in \mathcal{L}(\mu)$ on E

if $\mu = m$, we may just write

$f \in \mathcal{L}$

Remarks: If f is measurable, bounded on E & if $\mu(E) < \infty$ then $f \in \mathcal{L}(\mu)$ on E.

Exercises: (1) If $a \le f(x) \le b$ for $x \in E$

& $\mu(E) < \infty$

$$a\mu(E) \le \int_E f \, d\mu \le b\mu(E)$$

(2) f & $g \in \mathcal{L}(\mu)$ on E & $f \le g$ $\forall x \in E$

$$\Rightarrow \int_E f \, d\mu \le \int_E g \, d\mu$$

(3) If $\mu(E) = 0$ & f is measurable

$$\int_E f \, d\mu = 0$$

(4) If $f \in \mathcal{L}(\mu)$ on E & $A \in \mathcal{M}$, $f \in \mathcal{L}(\mu)$ on A.

Theorem:

a) Suppose f is measurable and non-negative on X.

For $A \in \mathcal{M}$ define $\phi(A) = \int_A f \, d\mu$

Then ϕ is countably additive on A

b) The same conclusion holds for $f \in \mathcal{L}(\mu)$ on X.

Proof: We will prove (a) for characteristic functions, then simple functions then non-neg. measurable $f^{\pm}$.

Once we have (a), (b) immediately follows since $f = f^+ - f^-$

Step 0: We have to show that

$$\phi(A) = \sum_{n=1}^{\infty} \phi(A_n) \quad \text{when } A_n\text{'s} \in \mathcal{M}$$

A_n's are disjoint & $A = \bigcup_{n=1}^{\infty} A_n$

Step 1: Characteristic functions

$$\phi(A) = \int_A K_E \, d\mu = \mu(A \cap E)$$

countably additive so ↘

Step 2: Simple functions $s(x) = \sum_{i=1}^{n} c_i K_{E_i}(x)$

$$\phi(A) = \sum_{i=1}^{n} c_i \mu(A \cap E_i)$$

also ↘ by countable additivity of μ.

Step 3: General non-neg measurable fns

Take any simple function $0 \le s \le f$

$$\int_A s \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} s \, d\mu \le \sum_{n=1}^{\infty} \phi(A_n)$$

(↑ by step 2; ↑ bec $s \le f$)

Taking the supremum over all such s (def'n of Lebesgue integral)

(1) $\phi(A) \le \sum_{n=1}^{\infty} \phi(A_n)$

Two cases: (1) $\phi(A_n) = +\infty$ for some A_n

$\Rightarrow \phi(A) = +\infty$ bec $\phi(A) \geq \phi(A_n)$

(2) $\phi(A_n) < \infty \quad \forall n$

Given any fixed $\varepsilon > 0$, we can pick

$0 \leq s \leq f$ with

$$\int_{A_1} s\,d\mu \geq \int_{A_1} f\,d\mu - \varepsilon = \phi(A_1) - \varepsilon$$

and (why?)

$$\int_{A_2} s\,d\mu \geq \int_{A_2} f\,d\mu - \varepsilon = \phi(A_2) - \varepsilon$$

So $\phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} s\,d\mu = \int_{A_1} s\,d\mu + \int_{A_2} s\,d\mu$

($0 \leq s \leq f$)

$$\geq \phi(A_1) - \varepsilon + \phi(A_2) - \varepsilon$$

$$\Rightarrow \phi(A_1 \cup A_2) \geq \phi(A_1) + \phi(A_2)$$

(take $\varepsilon \to 0$)

Repeating: $\phi(A_1 \cup A_2 \cup \dots \cup A_n) \geq \sum_{i=1}^{n} \phi(A_i)$
n-times

$$\Rightarrow \boxed{\phi(A) \geq \sum_{i=1}^{n} \phi(A_n)} \quad (2)$$

$A \supset \bigcup_{i=1}^{n} A_i$

(1) & (2) $\Rightarrow \phi(A) = \sum_{n=1}^{\infty} \phi(A_n)$.

∎

Corollary: If $A, B \in \mathcal{M}$, $B \subset A$ & $\mu(A \setminus B) = 0$, then

$$\int_A f \, d\mu = \int_B f \, d\mu$$

Proof: $\int_A f \, d\mu = \int_B f \, d\mu + \int_{A \setminus B} f \, d\mu$

$\phi(A)$ $\quad$ $\phi(B)$ $\quad$ $\phi(B \setminus A) = 0$ because $\mu(B \setminus A) = 0$

Important: Sets of measure zero don't have an effect on integration

So if $f = g$ except on a set of measure zero, then

1) (We say $f = g$ almost everywhere)

2) $\int_A f\,d\mu = \int_A g\,d\mu$

Theorem: If $f \in \mathcal{L}(\mu)$ on E, then ① $|f| \in \mathcal{L}(\mu)$ on E & ② $\left|\int_E f\,d\mu\right| \leq \int_E |f|\,d\mu$

Proof: ① Let $E^+ := \{x \in E \mid f(x) \geq 0\}$

$E^- := \{x \in E \mid f(x) < 0\}$

$E = E^+ \cup E^-$ $\quad \int_{E^+} f^+\,d\mu < \infty \quad \int_{E^-} f^-\,d\mu < \infty$

$\Rightarrow \int_E |f|\,d\mu = \int_{E^+} |f|\,d\mu + \int_{E^-} |f|\,d\mu \in \mathcal{L}(\mu)$

② $f \leq |f|$, $-f \leq |f|$

$\Rightarrow \int f \leq \int |f|$ & $-\int f \leq \int |f| \Rightarrow$

Lebesgue's Monotone Convergence Theorem

Let $E \in \mathcal{M}$ & $\{f_n\}_{n=1}^{\infty}$ be measurable

with $0 \le f_1(x) \le f_2(x) \le \dots \le \dots \quad \forall x \in E$

Let f: $f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in E$

Then $\int_E f\,d\mu = \lim_{n\to\infty} \int_E f_n\,d\mu$

Proof: Since the sequence is monotonic,

$f(x) = \sup_n f_n(x)$, hence measurable.

Moreover: $f_n \le f \Rightarrow$

$$\sup_n \int_E f_n\,d\mu \le \int_E f\,d\mu \qquad (1)$$

Now, let s be a simple function $0 \le s \le f$ and let $\alpha < 1$. Define

$$E_n = \{ x \in E \mid f_n(x) \ge \alpha s(x) \}$$

$\Rightarrow E_1 \subset E_2 \subset E_3 \ldots$ with $\bigcup_{n=1}^{\infty} E_n = E$

(because $\lim_{n\to\infty} f_n(x) = f(x) \ge s(x) > \alpha s(x)$)

So $\int_E f_n \, d\mu \ge \int_{E_n} f_n \, d\mu > \alpha \int_{E_n} s \, d\mu$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n \, d\mu \ge \lim_{n\to\infty} \int_{E_n} f_n \, d\mu > \lim_{n\to\infty} \alpha \int_{E_n} s \, d\mu$$

$\int_{E_n} s \, d\mu \to \phi(E_n)$ (countably additive)

$$\Rightarrow \lim_{n\to\infty} \int_E f_n \, d\mu \ge \alpha \int_E s \, d\mu \quad \left(\lim_{n\to\infty} \phi(E_n) = \phi(E) \right)$$

now, take $\alpha \to 1 \Rightarrow \lim_{n\to\infty} \int_E f_n \, d\mu \ge \int_E s \, d\mu$

Take a sup over $0 \le s \le f$

$$\Rightarrow \boxed{\lim_{n\to\infty} \int_E f_n \, d\mu \ge \int_E f \, d\mu} \quad (2)$$

• (1) & (2) $\Rightarrow$ ▨

Corollary: If $f_1, f_2 \in \mathcal{L}(\mu)$ on $E \in \mathcal{M}$ then $f_1 + f_2 \in \mathcal{L}(\mu)$ &

$$\int_E f_1 + f_2 \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu$$

Proof: ~~(I)~~ non-neg. fct.

Step 1: $f_1 = k_A$, $f_2 = k_B$

$$\int_E (f_1 + f_2)\, d\mu = \int_E (k_A + k_B)\, d\mu = \mu(E \cap A) + \mu(E \cap B)$$

$$= \int_E f_1 \, d\mu + \int_E f_2 \, d\mu$$

Step 2: Simple functions (same thing), ≥ 0

$$\int_E (f_1 + f_2) d\mu = \int_E \sum_{i=1}^{m} a_i k_{A_i} + \sum_{j=1}^{m} b_j k_{B_j} \, d\mu$$

$$= \cdots = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu$$

Step 3: $f_1 \geq 0$, $f_2 \geq 0$

Let $\{s_n\}$ & $\{t_n\}$ be monotonically increasing sequences of simple functions with $s_n \to f_1$, $t_n \to f_2$

(we know this is possible from a prev. thm)

Let $u_n = s_n + t_n$

$$\Rightarrow \int_E u_n \, d\mu = \int_E s_n \, d\mu + \int_E t_n \, d\mu$$

(by step 2)

$\Rightarrow$ by MCT we can take limits to get

$$\int_E f_1 + f_2 \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu$$

So now we have additivity for non-neg. fcts.

(II) $f_1 \geq 0$, $f_2 \leq 0$

Let $E^+ = \{x \in E \mid f_1(x) + f_2(x) \geq 0\}$

$E^- = \{x \in E \mid f_1(x) + f_2(x) < 0\}$

Since $f_2 \leq 0$, then $-f_2 \geq 0$ & by (I) we have

$$\int_{E^+} f_1 \, d\mu = \int_{E^+} f_1 + f_2 \, d\mu + \int_{E^+} -f_2 \, d\mu$$

(1) $\Longleftrightarrow$ $\boxed{\int_{E^+} f_1 \, d\mu + \int_{E^+} f_2 \, d\mu = \int_{E^+} f_1 + f_2 \, d\mu}$

Similarly $f_1 \geq 0$ & $-(f_1+f_2) \geq 0$ on E^-

So $\int_{E^-} f_1\,d\mu - \int_{E^-} f_1+f_2\,d\mu = -\int_{E^-} f_2\,d\mu$

$\Leftrightarrow$ $\int_{E^-} f_1\,d\mu + \int_{E^-} f_2\,d\mu = \int_{E^-} f_1+f_2\,d\mu$ (2)

(1) + (2) $\Rightarrow$ ↘

(III) Arbitrary measurable f_1, f_2

$E = E^{++} \cup E^{+-} \cup E^{-+} \cup E^{--}$

$E^{++} = \{x \in E \mid f_1(x) \geq 0, f_2(x) \geq 0\}$

Use (I) & (II) twice each ... and add.



Theorem: If $f(x) = \sum_{n=1}^{\infty} f_n(x)$ on $E \in \mathcal{M}$ where $\{f_n\}_{n=1}^{\infty}$ are non-negative measurable $f \pm$, then

$$\int_E f(x)\,d\mu = \sum_n \int_E f_n\,d\mu$$

Proof: $\sum_{n=1}^{m} f_n(x) =: g_m(x)$

$\{g_m\}$ is monotonically increasing

& $F(x) = \lim_{m\to\infty} g_m(x)$

$$\Rightarrow \int_E F\,d\mu = \lim_{m\to\infty} \int g_m\,d\mu$$

$$\int_E \sum_{n=1}^{\infty} f_n(x)\,d\mu = \lim_{m\to\infty} \int \sum_{n=1}^{m} f_n(x)\,d\mu$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} \int f_n(x)\,d\mu$$

$$= \sum_{n=1}^{\infty} \int f_n(x)\,d\mu \quad \blacksquare$$

Fatou's Lemma: Let $\{f_n\}$ be a sequence of non-negative measurable functions with

$F(x) := \liminf_{n\to\infty} f_n(x)$ $\quad x \in E \in \mathcal{M}$

then $\int_E F\,d\mu \le \liminf_{n\to\infty} \int_E f_n\,d\mu$

Proof: Let $g_n(x) = \inf_{i \geq n} f_i(x)$

$\Rightarrow g_n$ is measurable on E

and $0 \leq g_1 \leq g_2 \leq g_3 \leq \ldots$

& $\lim_{n\to\infty} g_n(x) = f(x)$ (def'n of liminf)

$\Rightarrow$ By MCT: $\boxed{\lim_{n\to\infty} \int_E g_n \, d\mu = \int_E f \, d\mu}$ (M)

But $g_n(x) \leq f_n(x)$ (g_n is the inf of a set containing f_n)

$\Rightarrow \int g_n \leq \int f_n$

$\Rightarrow \liminf_{n\to\infty} \int g_n \leq \liminf_{n\to\infty} \int f_n$

$\Rightarrow \int_E f \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu$

(M)

Course Evaluations:

Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions with

$$\lim_{n\to\infty} f_n(x) =: f(x) \quad \forall x \in E \in \mathcal{M}$$

Suppose $\exists g \in \mathcal{L}(\mu)$ on E with $|f_n| \leq g$

then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$$

Proof:

f_n measurable, $|f_n| \leq g \Rightarrow f_n^+ \leq g$ & $f_n^- \leq g$

$\Rightarrow f_n \in \mathcal{L}(\mu)$

& $f \in \mathcal{L}(\mu)$

We use Fatou's lemma on $g + f_n \geq 0$ & on $g - f_n \geq 0$

to get 1): $\int_E \liminf_{n\to\infty}(g+f_n) \le \liminf_{n\to\infty} \int_E g+f_n \, d\mu$

$\int_E g+f$ $\qquad$ $\int_E g + \liminf_{n\to\infty} \int_E f_n \, d\mu$

$\int_E g + \int_E f \le \int_E g + \liminf_{n\to\infty} \int_E f_n \, d\mu$ — (1)

2): $\int_E \liminf_{n\to\infty}(g-f_n) \le \liminf_{n\to\infty} \int_E g-f_n \, d\mu$

$\int_E g - \int_E f$ $\qquad$ $\int_E g - \liminf_{n\to\infty} \int_E f_n \, d\mu$

$\Rightarrow -\int_E f \, d\mu \le \liminf_{n\to\infty} \left[-\int_E f_n \, d\mu\right]$

$\Rightarrow \int_E f \, d\mu \ge \limsup_{n\to\infty} \int_E f_n \, d\mu$ — (2)

(1) & (2) $\Rightarrow$ ∎

Corollary: If $\mu(E) < \infty$, $\{f_n\}$ is uniformly bounded on E & $f_n \to f$ then $\int_E f = \lim_{n\to\infty} \int_E f_n \, d\mu$

Riemann Vs Lebesgue Integral

Theorem: If $f \in \mathcal{R}$ on $[a,b]$ then

(a) $f \in \mathcal{L}$ on $[a,b]$

and $\underbrace{\int_a^b f\,dx}_{\text{Lebesgue integral}} = \underbrace{\mathcal{R}\int_a^b f\,dx}_{\text{Riemann Integral.}}$

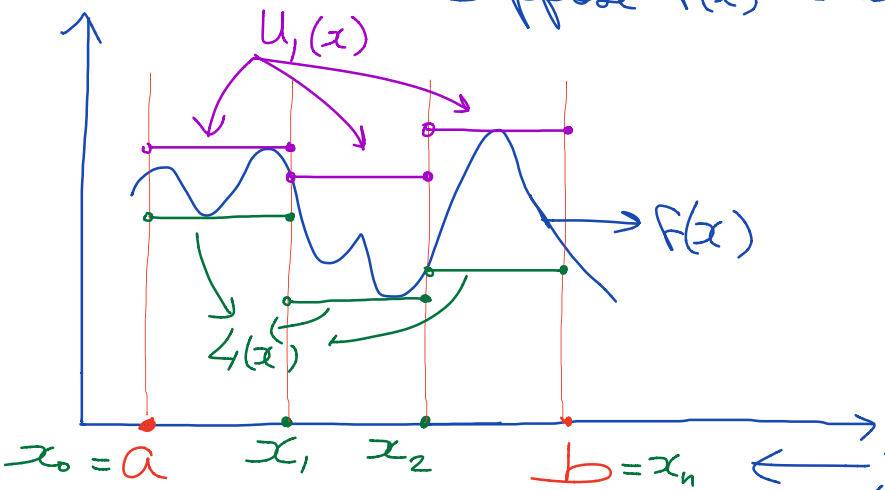
(b) If f is bounded on $[a,b]$ then $f \in \mathcal{R}$ if and only if it is continuous almost everywhere on $[a,b]$

Comment: (a) $\Rightarrow$ Lebesgue int is at least as "good" as the Riemann integral

(b) $\Rightarrow$ Riemann integral is limited.

Proof Sketch:

Suppose $f(x)$ is bounded



Refining the partition $\Rightarrow L_1(x) \leq L_2(x) \leq \cdots \leq f(x)$

and $f(x) \leq \cdots \leq U_2(x) \leq U_1(x)$

$L(x) := \lim_{k\to\infty} L_k(x)$, $U(x) = \lim_{k\to\infty} U_k(x)$

- bounded & measurable
- $L(x) \leq f(x) \leq U(x)$ — (1)
- $\int L\,dx = R\int f\,dx$, $\int U\,dx = R\int f\,dx$

($\lim_{k\to\infty}{}^{''} \int L_k$, and $\{L_k\}$ is non neg. & incr.)

So: $f \in R \iff \int L\,dx = \int U\,dx$ — (2)

but $L \leq U \Rightarrow$ (2) is possible iff $L = U$ a.e.

but by ① $L \le f \le U \Rightarrow f = L = U$ a.e.

$\Rightarrow f \in \mathcal{M}$ & $\int_a^b f\,dx = \mathcal{R}\int_a^b f\,dx$

(b). If $x \notin$ to any partition P_k then $U(x) = L(x)$ iff f is continuous at x.

• Remains to worry about $x \in P_k$ for some k. However each P_k is finite and $\bigcup_{k=1}^{\infty} P_k$ is countable, so $m\left(\bigcup_{k=1}^{\infty} P_k\right) = 0$

Thus $L(x) = U(x)$ a.e. $\Leftrightarrow$ f is cont. a.e.

$\Leftrightarrow f \in \mathcal{R}$.

Theorem: If $f \in \mathcal{L}$ on $[a\ b]$ & $F(x) = \int_a^x f\,dt$ then $F'(x) = f$ a.e. on E.

Converse: If F is differentiable at all $x \in [a,b]$ & if $F' \in \mathcal{L}$ on $[a,b]$ then

$F(x) - F(a) = \int_a^x F'(t)$

Functions of Class $\mathcal{L}^2$

Defn: $f \in \mathcal{L}^2(\mu)$ on $X \iff \int_X f^2 \, d\mu < +\infty$

Defn: If $f \in \mathcal{L}^2(\mu)$, $\|f\| = \left(\int_X |f|^2 \, d\mu \right)^{1/2}$

$\mathcal{L}^2_\mu$-norm of f

Theorem (Cauchy Schwarz):

$$\int_X |fg| \, d\mu \leq \|f\| \, \|g\|$$

Proof: If $g = 0$ or $f = 0$ ✓

otherwise: $0 \leq \int (|f| + \lambda |g|)^2 \, d\mu = \|f\|^2 + \lambda^2 \|g\|^2 + 2\lambda \int_X |fg| \, d\mu$

$\forall \lambda \in \mathbb{R}$

Take $\lambda = - \dfrac{\int_X |fg| \, d\mu}{\|g\|^2}$

$$\Rightarrow 0 \leq \|f\|^2 + \frac{\left[\int_X |fg| \, d\mu \right]^2}{\|g\|^4} \|g\|^2 - \frac{2 \left[\int_X |fg| \, d\mu \right]^2}{\|g\|^2}$$

$\Rightarrow$ ✓ ∎

Theorem (Δ-ineq.):

$$f \in \mathcal{L}^2(\mu) \ \& \ g \in \mathcal{L}^2(\mu) \Rightarrow \boxed{\|f+g\| \leq \|f\| + \|g\|}$$

Proof: $\|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2\int f g \, d\mu$

$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|$ (Cauchy Schwarz)

$= (\|f\| + \|g\|)^2$

∎

Remark: $\mathcal{L}^2(\mu)$ is a metric space with distance $\|f-g\|$ (once we identify functions that are equal a.e.)

Theorem: The Continuous functions are a dense subset of $\mathcal{L}^2 := \mathcal{L}^2(m)$ on $[a,b]$.

Proof: Need to show that $\forall f \in \mathcal{L}^2$ and any $\varepsilon > 0$, $\exists g$ continuous on $[a,b]$ s.t.

$$\|f-g\| < \varepsilon.$$

(I) Charact. fts.

Let A be a closed subset of $[a,b]$ and let

$$t_A(x) = \inf_{y \in A} |x-y|$$

Let $g_n(x) = \frac{1}{1 + n\,\ell_A(x)} \quad \forall x \in [a,b],\ n = 1,2,\ldots$

Note that $g_n(x)$ is continuous on $[a,b]$ (why)

with $g_n(x) = 1 \quad \forall x \in A$.

Moreover $g_n(x) \xrightarrow[n\to\infty]{} 0 \quad \forall x \in [a,b] \setminus A$

So $\lim_{n\to\infty} \|g_n - \mathbb{1}_A\| = \lim_{n\to\infty} \left[\int_{[a,b]\setminus A} g_n^2\, dx\right]^{1/2} = 0$ ①

(A closed; by the DCT)

(Characteristic functions of closed sets are well approximated in $\mathcal{L}^2$ by continuous functions)

But by regularity, for any $A \in \mathcal{M}(\mu)$

$\exists$ F, closed with $\mu(A \setminus F) < \varepsilon$

$\Rightarrow$ ① holds for all $A \in \mathcal{M}$

(II) Non-neg. Simple fcts

$\Rightarrow$ ① holds for simple functions (s instead of $\mathbb{1}_A$)

(III) Non-neg. Measurable f'ns.

$f \geq 0$

Let $f \in \mathcal{L}^2$ and denote by $\{s_n\}$, a sequence of monotincally inc simple f'ns with $s_n \geq 0$ & $\lim_{n\to\infty} s_n = f \geq 0$

$|f - s_n|^2 \leq |f|^2 \Rightarrow$ by the DCT $\int |f-s_n|^2 \to 0$

$\Rightarrow \searrow$

IV General f'ns

$f = f^+ - f^-$ ...

$\Rightarrow \searrow$

Orthonormal sets of functions:

$\{\phi_n\}$ is orthonormal if

$$\int_X \phi_n \bar{\phi}_m \, d\mu = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

So $\phi_n \in \mathcal{L}^2(\mu)$. For a function $f \in \mathcal{L}^2(\mu)$

let $c_n = \int_X f \bar{\phi}_n \, d\mu \quad (n = 1, 2, 3, \ldots)$

We write $f \sim \sum_{n=1}^{\infty} c_n \phi_n$

Theorem: $\forall f \in \mathcal{L}^2$, if $f \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$ (ϕ_n orthonormal)

then $\sum_{n=1}^{\infty} |c_n|^2 \leq \int_X |f|^2 \, d\mu$ and $\lim_{n\to\infty} c_n = 0$.

Proof: See thm 8.11, 8.12.

Theorem: (Parseval for Fourier series)

Let $f \in \mathcal{L}^2$ on $[-\pi, \pi]$ with $f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$

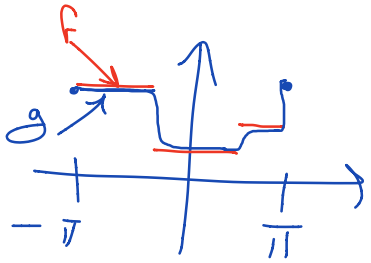
and let $s_n = \sum_{m=-n}^{n} c_m e^{imx}$

then $\|f - s_n\| \xrightarrow[n\to\infty]{} 0$ & $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 dx$

Proof: Cont's fct's are dense in $\mathcal{L}^2$ on $[-\pi, \pi]$ $\Rightarrow$

$\forall \varepsilon > 0, \exists$ cont's g : $\|f - g\| < \varepsilon/2$

Pick g, so that $g(\pi) = g(-\pi)$



$\Rightarrow g$ can be extended to a periodic cont's fct.

$\Rightarrow \exists$ a trig. polynomial T_ε of degree N_ε s.t.

$$\|g - T\| < \varepsilon/2$$

$$\Rightarrow \text{by Thm 8.11 } (L^2 \text{ version}) \text{ \& } \|S_n - f\| \leq \|T - f\| < \varepsilon$$

$$n \geq N_\varepsilon$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \|S_n - f\| = 0}$$

To show that $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 dx$

note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(f)\,\bar{f}\,dx = \sum_{-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\bar{f}(x)\,dx$$

$$= \sum_{-N}^{N} c_n \bar{c}_n$$

and $\left|\int |f|^2 - \int S_N(f)\bar{f}\right| \leq \int \left(|f - S_N(f)|\,|\bar{f}|\right) d\mu$

$$\leq \left(\int (f - S_N)^2 \int |f|^2\right)^{1/2} d\mu$$

Cauchy-Schwarz

$$\xrightarrow[N\to\infty]{} 0$$

Corollary: $f \in L^2$ on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} f e^{-inx}\,dx = 0 \Rightarrow \|f\| = 0$

(Two functions with the same Fourier series differ on a set of measure 0)

Convergence in $\mathcal{L}^2(\mu)$: Suppose $f \in \mathcal{L}^2(\mu)$ & $f_n \in \mathcal{L}^2(\mu)$.

We say $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$ if $\lim_{n\to\infty} \|f_n - f\| = 0$

- We say $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$ if $\forall \varepsilon > 0$ there is an integer N s.t. $n \geq N, m \geq N \Rightarrow \|f_n - f_m\| \leq \varepsilon$

Theorem: If $\{f_n\}$ is a Cauchy Sequence in $\mathcal{L}^2(\mu)$, then

$\exists f \in \mathcal{L}^2(\mu)$ s.t. $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$ ($\mathcal{L}^2(\mu)$ is a complete metric space)

Proof:

$\{f_n\}$ is Cauchy $\Rightarrow \exists \{n_k\}$, $k = 1, 2, 3, \ldots$ s.t.

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k}$$

Let $g \in \mathcal{L}^2(\mu) \Rightarrow \int_X |g(f_{n_k} - f_{n_{k+1}})|\, d\mu \leq \|g\| \, \|f_{n_k} - f_{n_{k+1}}\|$

$$\leq \frac{\|g\|}{2^k}$$

$$\Rightarrow \sum_{k=1}^{\infty} \int |g(f_{n_k} - f_{n_{k+1}})|\, d\mu \leq \|g\|$$

Sequence of non-neg measurable fts $\Rightarrow$ can exchange order of $\sum$ & $\int$

$$\Rightarrow \int \sum_{k=1}^{\infty} |g(f_{n_k} - f_{n_{k+1}})|\, d\mu \leq \|g\| < \infty \Rightarrow |g(x)| \sum_{k=1}^{\infty} |(f_{n_k} - f_{n_{k+1}})| < \infty \text{ a.e.}$$

$\Rightarrow \underbrace{\sum_{k=1}^{\infty} |(f_{n_k} - f_{n_{k+1}})|}_{k^{th} \text{ partial sum: } f_{n_{k+1}}(x) - f_{n_1}(x)} < +\infty$ a.e. (why?) $\Rightarrow$ converges a.e.

$\Rightarrow f(x) := \lim_{k\to\infty} f_{n_k}(x)$ defines f a.e. on X

Remains to show this is the f described in the theorem.

Let $\varepsilon > 0$ & N_ε : $n_i, n_k > N_\varepsilon \Rightarrow \boxed{\|f_{n_i} - f_{n_k}\| \leq \varepsilon}$ — (*)

$\Rightarrow \int_X \liminf_{i\to\infty} |f_{n_i} - f_{n_k}|^2 \, d\mu \leq \liminf_{i\to\infty} \int_X |f_{n_i} - f_{n_k}|^2 \, d\mu$

$$\|f - f_{n_k}\| \leq \liminf_{i\to\infty} \|f_{n_i} - f_{n_k}\| \leq \varepsilon \quad \text{by (*)}$$

$\Rightarrow f - f_{n_k} \in \mathcal{L}^2(\mu) \Rightarrow f \in \mathcal{L}^2(\mu)$

taking $\varepsilon \to 0$: $\lim_{k\to\infty} \|f - f_{n_k}\| = 0$

and $\|f - f_n\| \leq \|f - f_{n_k}\| + \|f_{n_k} - f_n\|$

($\|f - f_{n_k}\| \to 0$, $\|f_{n_k} - f_n\| \to 0$)

$\Rightarrow \|f - f_n\| \to 0 \Rightarrow \{f_n\}$ converges to $f \in \mathcal{L}^2(\mu)$.

The Riesz Fischer Theorem

Let $\{\phi_n\}$ be orthonormal on X. Suppose

$\sum |c_n|^2$ converges & let $s_n = \sum_{i=1}^{n} c_i \phi_i$.

There exists $f \in \mathcal{L}^2(\mu)$ s.t. $\{s_n\}$ converges to f in $\mathcal{L}^2(\mu)$

and $f \sim \sum_{n=1}^{\infty} c_n \phi_n$

Proof: Suppose $n > m$: $\|s_n - s_m\|^2 = \sum_{i=m+1}^{n} |c_i|^2$ (Parseval)

$\Rightarrow \{s_n\}$ is cauchy in $\mathcal{L}^2(\mu) \Rightarrow \exists f \in \mathcal{L}^2(\mu) : \lim_{n\to\infty} \|f - s_n\| = 0$

Let $n > k$: $\int_X f \bar{\phi}_k \, d\mu - c_k = \int f \bar{\phi}_k \, d\mu - \int s_n \bar{\phi}_k \, d\mu$ (by orth. of ϕ_k's)

$\Rightarrow$

$$\left| \int_X f \bar{\phi}_k \, d\mu - c_k \right| \leq \|f - s_n\| \, \|\phi_k\|$$

take $n \to \infty$

$\Rightarrow$ RHS $\to 0$, so $\boxed{\int f \bar{\phi}_k \, d\mu = c_k}$

∎

Defn: An orthonormal set $\{\phi_n\}$ is <u>complete</u> if

$\forall f \in \mathcal{L}^2(\mu)$: $\int f \bar{\phi}_n \, d\mu = 0 \ \forall n \Rightarrow \|f\| = 0$

Theorem: (Complete orthonormal set $\Rightarrow$ Parseval holds)

Let $\{\phi_n\}$ be a complete orthonormal set

If $f \in \mathcal{L}^2(\mu)$ & $f \sim \sum_{n=1}^{\infty} c_n \phi_n$

then $\int_X |f|^2 d\mu = \sum_{n=1}^{\infty} |c_n|^2$

Proof: Since $\sum_{n=1}^{\infty} |c_n|^2 \leq \int_X |f|^2 d\mu$ (Bessel)

then $\sum |c_n|^2$ converges.

$\Rightarrow$ By Riesz-Fischer: $\exists g \in \mathcal{L}^2(\mu)$

s.t. $g \sim \sum c_n \phi_n$ & $\|g - S_n\| \to 0$

$S_n := \sum_{i=1}^{n} c_i \phi_i$

$\Rightarrow \|S_n\| \to \|g\|$

But $\|S_n\|^2 = \sum_{i=1}^{\infty} |c_i|^2$ (orth. of ϕ_n's)

$\Rightarrow \int |g|^2 d\mu = \sum_{n=1}^{\infty} |c_n|^2$

$g \sim \sum_{n=1}^{\infty} c_n \phi_n$

but $f \sim \sum_{n=1}^{\infty} c_n \phi_n$

$\Rightarrow \int (f-g)\overline{\phi_n}\, d\mu = 0 \Rightarrow \|f-g\| = 0$ by completeness

$\Rightarrow \|f\|^2 = \|g\|^2 = \sum |c_n|^2$

Recap

Let $\{\phi_n\}$ be an an orthonormal set on X:

Riesz-Fischer

$$\sum |c_n|^2 \text{ converges} \Rightarrow \begin{cases} \sum_{i=1}^{n} c_i \phi_i \xrightarrow[n\to\infty]{} f \in \mathcal{L}^2 \\ f \sim \sum_{n=1}^{\infty} c_n \phi_n \end{cases}$$

Last theorem:

$$\left.\begin{array}{l} \phi_n \text{ is complete} \\ f \in \mathcal{L}^2(\mu) \text{ and } f \sim \sum_{n=1}^{\infty} c_n \phi_n \end{array}\right\} \Rightarrow \int |f|^2 d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$